Determination of an Operator Algebra for the Two-Dimensional Ising Model

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A previous publication showed how the critical indices for the two-dimensional Ising model could be derived from an assumed form of an operator algebra which describes how the product of two fluctuating variables may be reduced to a linear combination of the basic fluctuating variables. In this paper, the previously used algebra is derived from the Onsager solution of the two-dimensional Ising model. The calculation makes use of a "disorder" variable which is mathematically the result of applying the Kramers-Wannier transformation to the Ising-model spin variable. The average of products of spin and disorder variables are evaluated at the critical point for the special case in which all the variables lie on a single straight line. The ordering of these variables on the line determines a "quantum number" Γ such that the average is nonzero only for $\Gamma=0$. Composition rules for this quantum number are derived and used to develop an algebra for the multiplication of complex variables at the critical point. Arguments are given to suggest the identifications of elements of the algebra as the spin, the energy density, the Kaufman spinors, and a stress density. The result of this calculation is the operator algebra which formed the starting point of the previous paper.

I. INTRODUCTION

Critical-point fluctuations can be described in terms of correlations among a set of local fluctuation variables, $O_\gamma(\vec{r})$. This set will include, for example, a local energy density and a local order parameter, as well as several other quantities which have fluctuations on a large spatial scale. The concept of reducibility[1-4] starts from the suggestion that the set $O_\gamma(\vec{r})$ might include only a finite number of fluctuation variables with really different large-scale fluctuations. If this is true, then any product

$$X=\prod_{i=1}^{n} O_{\gamma_i}(\vec{r}_i)$$

of variables within some small neighborhood must be reducible to a linear combination of the basic operators

$$X=\sum_\gamma A_\gamma O_\gamma(\vec{R}) \quad ,$$
$$\vec{R}=(1/n)\sum_i \vec{r}_i \quad .$$

There the A_γ's are a set of numbers which can depend upon the γ_i and the values of the spacial differences $\vec{r}_i-\vec{r}_j$. In particular, the reducibility hypothesis suggests that a product of two nearby basic operators is reducible as

$$O_\alpha(\vec{r}_1)\,O_\beta(\vec{r}_2)=\sum_\gamma A_{\alpha\beta,\gamma}(\vec{r})\,O_\gamma(\vec{R}) \quad ,$$
$$\vec{r}=\vec{r}_1-\vec{r}_2 \quad , \quad R=\tfrac{1}{2}(\vec{r}_1+\vec{r}_2). \qquad (1.1)$$

The coefficients A in the reducibility relations describe a kind of algebra for critical fluctuations. We believe that an understanding of the structure of this algebra is a powerful tool which perhaps might be used for predicting critical indices. This belief is based upon the result of a previous calculation,[2] in which all critical indices for the two-dimensional Ising model were derived from: (a) a listing of the O_γ, (b) the scaling concept, (c) symmetry properties, and (d) a knowledge of which of the coefficients $A_{\alpha\beta,\gamma}$ were nonvanishing.

The present paper is devoted to defining the reduction algebra for the two-dimensional Ising model so as to exhibit the justification for the assumptions used in Ref. 2. We begin by introducing, in Sec. II, a new fluctuating variable $\mu_{\vec{r}}$, which roughly represents the amount of disorder in the neighborhood of the point $\vec{r}$. More precisely, this new variable is the transform of the standard local magnetization variable $\sigma_{\vec{r}}$ under the Kramers-Wannier[5] transformation. This transformation describes an important symmetry of the Ising model which is very useful in determining symmetries in the coefficients A.

Section III lists the important fluctuation variables O_γ. In addition to $\sigma_{\vec{r}}$ and $\mu_{\vec{r}}$, this list includes the energy density and the components of a stress tensor t_{ij}. Kawasaki[6] has used this tensor for discussions of the liquid-gas transition, but its importance in the Ising model has not been emphasized in the past. The remaining variables on this list are the two-component spinor or "fermion" variables used by Kaufman,[7] by Schultz, Mattis, and Lieb,[8] and by one of the present authors.[9]

In Sec. IV, we consider correlations among the basic variables for the special case in which the variables all lie on a single straight line. Previous calculations of spin correlations[1] have shown that special simplicities ensue for this case. Even more remarkable symmetry relations appear when we consider correlations of products of σ's and μ's

along a line. The ordering of the operators determines a "quantum number" Γ, where 2Γ is an integer. The value of Γ, which is in appearance similar to an angular momentum label, determines whether or not the correlation vanishes. By using operators D_γ each of which has a well-defined value of Γ, we can make the reduction relation (1.1) take a particularly simple form. Furthermore, the critical-point correlations of any product of D_γ's on the line is found and written explicitly.

Section V lists the coefficients in the reduction formulas (1.1).

All the calculations of the present paper are, of course, derived from the Onsager solution[10] of the Ising model. However, the main results are sets of symmetry relations for $A_{\alpha\beta,\gamma}$ which might just have been guessed without the Onsager solution. Hence, similar symmetries might be found for other critical fluctuation problems. If so, then the methods of Ref. 2 could perhaps provide a technique for finding critical indices from first principles.

II. DISORDER VARIABLES

A. Magnetic Dislocations

At zero magnetic field, the Ising model has a partition function which can be expressed as a sum over all spins according to

$$Z\{K\} = \sum_{\{\sigma_{jk}\}=\pm 1} e^{G\{K,\sigma\}} \quad , \tag{2.1}$$

$$G\{K,\sigma\} = \sum_{j,k} \sigma_{j,k}\left[K_x\left(j+\tfrac{1}{2},k\right)\sigma_{j+1,k} + K_y\left(j,k+\tfrac{1}{2}\right)\sigma_{j,k+1}\right].$$

Here we have a lattice in which j increases in the x direction, k increases in the y direction, and we have allowed all the coupling constants to be unequal. A basic cell in this lattice is depicted in Fig. 1(a).

Eventually, we shall wish to set all the coupling constants equal to one another. However, before doing this, we introduce a magnetic dislocation into the lattice by letting[11]

$$K \to -K \tag{2.2}$$

for all the coupling constants along the path indicated in Fig. 1(b). Then, imagine that all the coupling constants indicated by light bars, are equal and positive, whereas those with heavy bars are equal to minus the others. Since there is an energetic advantage for spins connected by light bars to be equal and spins connected by heavy bars to be opposite, this reversal of coupling constants tends to introduce a Bloch wall into the spin system. If $\{K\}$ is the set of coupling constants before the reversal and $\{\tilde{K}\}$ the set after, we define a correlation function

$$\langle \mu_{r_1}\mu_{r_2}\rangle\{K,\Gamma\} = Z\{\tilde{K}\}/Z\{K\} \quad . \tag{2.3}$$

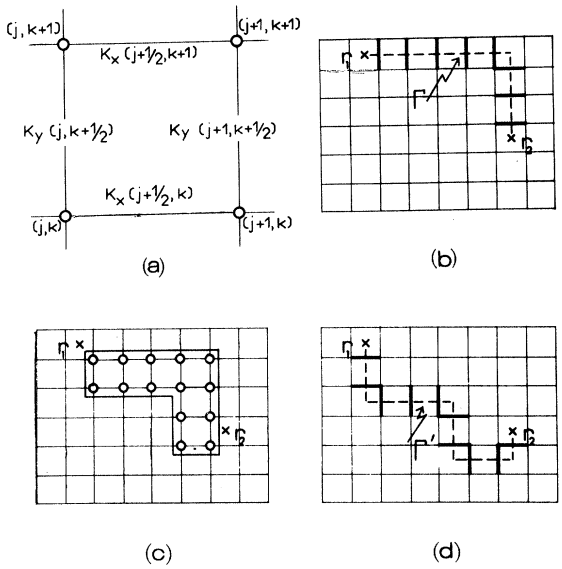


FIG. 1. (a) A basic cell of the model used in the present work. All the coupling constants K_x and K_y are different. (b)–(d) The μ variables are denoted symbolically by a cross (×). Heavy bars indicate the bonds subject to the transformation $K \to -K$. The correlation function involving two μ variables is path independent. We pass from the path Γ in (b) to the path Γ' in (d) simply by changing the sign of the spins inside the region outlined in (c). The partition function is invariant under such a change, whereby the path independence follows.

The right-hand side of (2.3) is physically an exponential of minus the cost in free energy for introducing a magnetic dislocation between $\vec{r}_1$ and $\vec{r}_2$ along the path Γ.

The result is independent of the path Γ. To see this path independence, consider the effect upon $Z\{\tilde{K}\}$ of changing the sign of a group of the dummy summation variables $\sigma_{j,k}$ in Eq. (2.1). Let us flip the signs of all the σ's at the positions circled in Fig. 1(c). Since the K's appear in the form $K\sigma\sigma'$, the effect of changing the sign of these summation variables is to flip the sign of all coupling constants connecting the region outlined with the remainder of the lattice. Under this transform

$$Z\{\tilde{K}\} \to Z\{\tilde{K}'\} \quad ,$$

where $\tilde{K}'$ no longer has minus signs on the path Γ but now has them on the path Γ' shown in Fig. 1(d). Since the change of a summation variable does not change the result of the sum,

$$Z\{\tilde{K}\} \equiv Z\{\tilde{K}'\} \quad .$$

Consequently, the correlation function defined in Eq. (2.3) has equal values for all possible paths connecting the points $\vec{r}_1$ and $\vec{r}_2$.

In a similar way, we can define correlations of any even number of μ_r's, as depicted, for example,

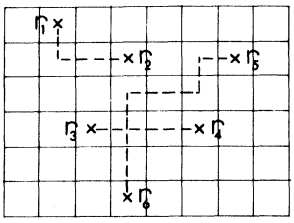


FIG. 2. The correlation function of several μ's is defined by a straightforward generalization of the case of two μ variables. The paths shown are arbitrary, because of the path independence.

in Fig. 2. These correlation functions are defined with the aid of paths through the lattice. But the result is path independent.

In just the same way as σ_r describes the order in the lattice, μ_r describes the disorder. At infinite temperatures, all K's vanish so that $Z\{\tilde{K}\} = Z\{K\}$. Hence, in the completely disordered system, $\langle \mu_r \mu_{r'} \rangle = 1$, for all r, r'.

Introducing $\langle \mu \rangle^2$ as

$$\langle \mu \rangle^2 = \lim_{|r-r'| \to \infty} \langle \mu_r \mu_{r'} \rangle \quad ,$$

we have

$$\langle \mu \rangle^2 = 1 \ , \quad T = \infty \ . \tag{2.4a}$$

For $T < T_c$, a long spin dislocation costs a "surface" free energy equal to the "surface" tension t times the length of the path.[11] Hence, for $T < T_c$,

$$\lim_{|r-r'| \to \infty} \ln \langle \mu_r \mu_{r'} \rangle \sim - |r - r'| t \ , \quad T < T_c \ . \tag{2.4b}$$

Therefore when there is long-range order, for $T < T_c$, the long-range disorder vanishes. If μ_r is directly analogous to σ_r,[12] we might expect that $\langle \mu \rangle^2$ decreases and goes to zero as T is reduced from infinity to T_c. To see this property, we must calculate correlations of the μ_r. The easiest way of doing this calculation is to employ the Kramers-Wannier transformation to relate μ correlations to σ correlations.

B. Path Formulation of Spin Correlations

To make this relation, one must introduce a path formulation for spin correlations. Consider a path Γ on the lattice as shown in Fig. 3(a). Take a new set of coupling constants $\{K'\}$ such that

$$K' = \begin{cases} K & \text{off the path} \\ K + i\tfrac{1}{2}\pi & \text{on the path} \end{cases} \ . \tag{2.5}$$

Notice that if K lies on the path, the structure $e^{K\sigma\sigma'}$, which appears in $Z\{K\}$, transforms according to

$$e^{K\sigma\sigma'} \to e^{K'\sigma\sigma'} = e^{K\sigma\sigma'} i\sigma\sigma' \ .$$

Therefore, in an n-step path,

$$Z\{K'\} = \sum_{\{\sigma\}} e^{G\{K',\sigma\}} = \sum_{\{\sigma\}} e^{G\{K,\sigma\}} \prod_{j=1}^{n} (i\,\sigma_j \sigma_{j+1}) \quad ,$$

where σ_j is the spin at the beginning of the jth step and σ_{j+1} is the spin at the end of it. Since $\sigma_j^2 = 1$, and since all spins, save the first and last, appear twice,

$$\frac{Z\{K'\}}{Z\{K\}} = \sum_{\{\sigma\}} e^{G\{K,\sigma\}} i^n \sigma_{r_1}\sigma_{r_2} \Big/ \sum_{\{\sigma\}} e^{G\{K,\sigma\}} = i^n \langle \sigma_{r_1}\sigma_{r_2} \rangle \quad .$$

As a result, the spin-spin correlation function is

$$\langle \sigma_{r_1}\sigma_{r_2} \rangle \{K\} = (-i)^n \, Z\{K'\}/Z\{K\} \ . \tag{2.6}$$

The results for the two kinds of correlations can be written in an even more symmetrical fashion in terms of a symmetric partition function $Y\{K\}$ defined by

$$Y\{K\} = Z\{K\}\ 2^{-\mathfrak{N}/2} \prod_{j,k} [\cosh 2K_x (j + \tfrac{1}{2}, k) \times \cosh 2K_y (j, k + \tfrac{1}{2})]^{-1/2} \quad , \tag{2.7}$$

where $\mathfrak{N}$ is the number of spins in the lattice. Since $\cosh 2K$ is invariant under $K \to -K$ and changes sign under $K \to K + \tfrac{1}{2}\pi i$, the results (2.3) and (2.6) may be expressed as

$$\langle \mu_{r_1}\mu_{r_2} \rangle \{K\} = Y\{\tilde{K}\} / Y\{K\} \quad , \tag{2.8a}$$

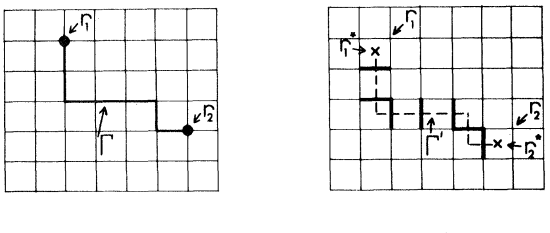
$$\langle \sigma_{r_1}\sigma_{r_2} \rangle \{K\} = Y\{K'\}/Y\{K\} \ . \tag{2.8b}$$

Here, as before, K has an opposite sign to K along Γ and is identical to K elsewhere; K' is $K + \tfrac{1}{2}\pi i$ along Γ' and equals K everywhere else.

C. Kramers-Wannier (K-W) Transform

The introduction of μ_r was motivated by the existance of an exact symmetry relating the $T > T_c$ and the $T < T_c$ regions in the two-dimensional Ising model. To describe this symmetry, define a function of K by

$$K^*(K) = \tfrac{1}{2} \operatorname{arcsinh} \left(\frac{1}{\sinh 2K} \right) \tag{2.9a}$$



(a) (b)

FIG. 3 Spins are denoted by a dot (•); μ's by a cross (×). Under the K-W transform the path Γ in (a) becomes the path Γ' shown in (b). After setting all the coupling constants equal, we find

$$\langle \sigma_{r_1}\sigma_{r_2} \rangle_{\Gamma} (\epsilon, u) = \langle u_{r_1}^* u_{r_2}^* \rangle_{\Gamma'} (-\epsilon, u).$$

or

$$\sinh 2K^*(K) = \frac{1}{\sinh 2K} \quad . \tag{2.9b}$$

Let us define a new set of coupling constants $\{K^*\{K\}\}$ by

$$\begin{aligned} K_x^*(j+\tfrac{1}{2},k) &= K^*[K_y(j+1,k+\tfrac{1}{2})] \quad , \\ K_y^*(j,k+\tfrac{1}{2}) &= K^*[K_x(j+\tfrac{1}{2},k+1)] \quad . \end{aligned} \tag{2.10}$$

The K-W transform corresponds to $K \to K^*(K)$ in $Y\{K\}$ and in correlation functions. Notice that at low temperatures the original coupling constants K are large but the new ones K^* are small. Conversely, at high temperatures where the K's are small, the K^*'s are large. Hence, the transformation (2.9) and (2.10) interchanges high and low temperatures.

The statement of the basic symmetry between high and low temperatures is very simple: For all possible $\{K\}$,

$$Y\{K^*\{K\}\} \equiv Y\{K\} \quad . \tag{2.11}$$

This result is proved in Appendix A.

To see one consequence of the symmetry (2.11), consider the homogeneous case in which all the K_x's are equal to each other and all the K_y's are the same also, but K_x is not necessarily equal to K_y. Then, $Y\{K\}$ can be written as a function of $\sinh 2K_x$ and $\sinh 2K_y$:

$$Y\{K\} = Y(\sinh 2K_x,\ \sinh 2K_y) \quad .$$

Then, the symmetry (2.11) implies that for large systems

$$Y(\sinh 2K_x,\ \sinh 2K_y) = Y\left(\frac{1}{\sinh 2K_y},\ \frac{1}{\sinh 2K_x}\right)$$

or

$$Y(a,b) = Y(b^{-1},\ a^{-1}) \quad .$$

Moreover, since there is no distinction between the x and y directions, $Y(a,b) = Y(b,a)$. As a result of these symmetries, Y can be considered to be a function of ϵ^2 and $\frac{1}{2}(u+u^{-1})$, where u is the asymmetry parameter

$$u^4 = a/b = \sinh 2K_x/\sinh 2K_y \quad , \tag{2.12}$$

such that $u - u^{-1}$ measures the asymmetry between the x and y directions, and ϵ is

$$\begin{aligned} -\epsilon &= \tfrac{1}{4}(a+b-a^{-1}-b^{-1}) \\ &= \tfrac{1}{4}[\sinh 2K_x + \sinh 2K_y - (\sinh 2K_x)^{-1} - (\sinh 2K_y)^{-1}] \quad . \end{aligned} \tag{2.13}$$

Near T_c, ϵ is proportional to $(T-T_c)/T_c$, so that ϵ measures the deviation from T_c.

Under the K-W transform, $\epsilon \to -\epsilon$ and $u \to u$. Under the transform which interchanges x and y, $u \to u^{-1}$ and $\epsilon \to \epsilon$. The result of the K-W transform is that, except for a trivial multiplicative term, the partition function is an even function of ϵ. Hence, the singular term in the energy $\sim \partial \ln Z/\partial\epsilon$ is odd in ϵ, while the singular term in the specific heat is even in ϵ.

Now apply the K-W transform to the spin-spin correlation function shown in Fig. 3(a). Let K^* be the transform of the set of coupling constants K, and K'^* be the transform of the set K' on the path Γ'. The transformed path is shown in Fig. 3(b). From Eqs. (2.8b) and (2.11) we find

$$\langle \sigma_{r_1}\sigma_{r_2}\rangle\{K\} = \frac{Y\{K'\}}{Y\{K\}} \equiv \frac{Y\{K'^*\}}{Y\{K^*\}} \quad . \tag{2.14}$$

But, the transformation operations act differently upon K and K^*. According to Eq. (2.10),

$$K \to K + \tfrac{1}{2}\pi i$$

implies

$$K^*(K) \to -K^*(K) \ , \tag{2.15a}$$

whereas

$$K \to -K$$

implies

$$K^*(K) \to K^*(K) + \tfrac{1}{2}\pi i \quad . \tag{2.15b}$$

Hence, we evaluate the ratio on the right-hand side of Eq. (2.14) by employing either one of the paths shown in Figs. 3(a) or 3(b). On the path Γ', K'^* is $-K^*$. Therefore,

$$Y\{K'^*\}/Y\{K^*\} = \langle u_{r_1}^* u_{r_2}^*\rangle\{K^*\} \quad , \tag{2.16}$$

where

$$r_1^* = (j_1^*, k_1^*) = (j_1 - \tfrac{1}{2},\ k_1 - \tfrac{1}{2}) \quad . \tag{2.17}$$

When we set all the coupling constants equal, we then find from (2.14) and (2.16) that

$$\langle \sigma_{r_1}\sigma_{r_2}\rangle(\epsilon,u) \equiv \langle u_{r_1}^* u_{r_2}^*\rangle(-\epsilon,u) \quad . \tag{2.18}$$

When r_1 and r_2 are far separated, we find

$$\langle \sigma_r\rangle(\epsilon,u) = \pm\langle \mu_r\rangle(-\epsilon,u) \quad , \tag{2.19}$$

so that as the temperature is reduced from ∞, the $\langle \mu_r\rangle$ decreases and approaches zero as $(T-T_c)^{1/8}$ in the neighborhood of the critical point.

The K-W transformation can be reduced to a set of simple rules. All correlation functions are invariant under the simultaneous replacement: $\epsilon \to -\epsilon$, $u \to u$, $\sigma \to \mu$, $\mu \to \sigma$, and $r = (j,k) \to r^* = (j-\frac{1}{2}, k-\frac{1}{2})$.

D. Correlation Functions Involving both μ's and σ's

The definition of correlation functions involving both μ's and σ's is very straightforward. Consider the product of two μ's and two σ's shown in Fig.

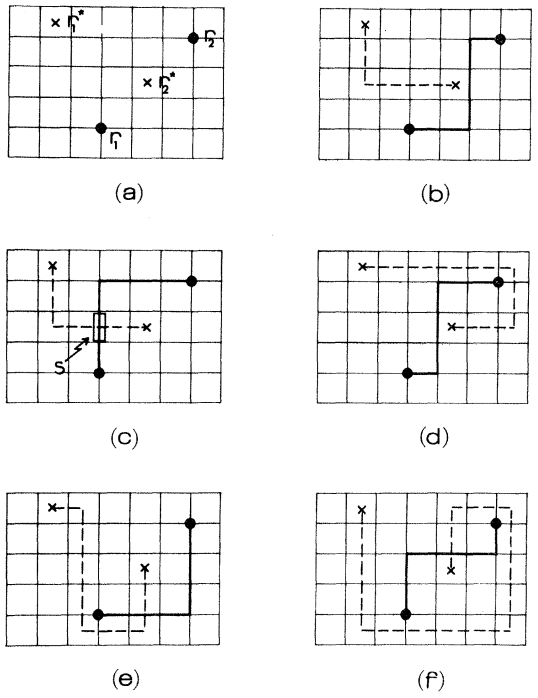


FIG. 4. In the cases involving both σ's and μ's only the *absolute value* of the correlation function is path independent, because a μ path crossing a σ variable produces an over-all change of sign. In the text it is proved that the correlations calculated with the paths shown in (b), (c), and (f) are the same, and have opposite sign to the ones in (d) and (e). The crossing S shown in (c) illustrates a case where there is a basic indeterminacy of sign produced by the noncommutativity of the operations defining σ and μ paths.

4(a). To define the correlation function, employ the paths shown in Fig. 4(b) and write

$$\tilde{K}=\begin{cases} K+\frac{1}{2}\pi i & \text{on} _____ \\ -K & \text{on} - - - - - \\ K & \text{elsewhere} , \end{cases} \tag{2.20a}$$

so that

$$\langle \mu_{r_1}\mu_{r_2}\sigma_{r_1'}\sigma_{r_2'}\rangle = Y\{\tilde{K}\}/Y\{K\} \quad . \tag{2.20b}$$

There is only one further difficulty. Notice that the paths shown in Fig. 4(b) do not intersect. This is important because the operations

$$K \to K+\tfrac{1}{2}\pi i \quad , \tag{2.21a}$$

$$K \to -K \tag{2.21b}$$

do not commute. If (2.21a) comes before (2.21b), then

$$e^{K\sigma\sigma'} \to e^{-K\sigma\sigma' i\sigma\sigma'} \quad , \tag{2.22a}$$

whereas if the order is reversed,

$$e^{K\sigma\sigma'} \to e^{-K\sigma\sigma'}(-i\sigma\sigma') \quad . \tag{2.22b}$$

For this reason, we always imagine that all operations (2.21a) are performed before all operations (2.21b). This assumption defines the meaning of intersecting paths like those shown in Fig. 4(c).

Nonetheless, the resulting correlation functions are not quite path independent. No possible deformations of σ paths can cause minus signs. Therefore, it is quite irrelevant whether we draw these paths, as in Fig. 4(b), or leave them out. However, the results are not quite independent of the μ path. The proof of μ-path independence fails if the correlation function includes a σ_r. Consider that one of the circled points in Fig. 1(c) contained a σ variable. Then as the path was changed from Γ to Γ', the change in sign of the spin variable would produce an over-all sign change in the correlation function.

This result can be reduced to a rule: As a μ path is deformed through a spin variable, the correlation function changes sign. There is no other path dependence. According to this rule, the correlation function defined by Fig. 4(d) or 4(e) is opposite in sign to the correlation function in Fig. 4(b) or 4(c) whereas the correlation function in Fig. 4(f) is the same as Fig. 4(b) or 4(c).

III. OTHER BASIC VARIABLES

A. General Considerations

So far, we have discussed in detail two important fluctuation variables, the local order variable σ_r and the disorder variable μ_r. We call any function of these variables an operator. There are two important kinds of operators: extensive and intensive. An *intensive operator* $x(r)$ depends only upon spin and disorder variables within a few lattice constants of the point $\vec{r}$; an *extensive operator* X is the sum over the entire lattice of an intensive operator:

$$X=\sum_r x(r) \quad . \tag{3.1}$$

Typical extensive operators are the Hamiltonian $\mathcal{H}$ and the total magnetization; their intensive partners are the energy density and the local magnetization σ_r.

Fluctuations in extensive operators are closely connected with thermodynamic derivatives. To see this, consider the effect of adding to the Hamiltonian at the critical point a set of terms of the form $x_\alpha X_\alpha$, where x_α are parameters and the X_α are operators. In symbols,

$$-\mathcal{H}/kT \to -\mathcal{H}/kT+\sum_\alpha x_\alpha X_\alpha \quad .$$

An added term of the form

$$x_\alpha X_\alpha = (\delta T/T_c)\mathcal{H}/kT_c$$

would effectively change the temperature from the critical value T_c to $T=T_c+\delta T$, whereas a term

$$x_\alpha X_\alpha = (\mu H/kT_c)\, M$$

would represent the addition of a magnetic field H. Then the derivative of any physical quantity $\langle Y\rangle$ with respect to one of the parameters would have the form

$$\frac{\partial}{\partial x_\alpha}\langle Y\rangle = \frac{\partial}{\partial x_\alpha}\sum_{\{\sigma_r\}=\pm 1} e^{-H/kT} Y \Big/ \sum_{\{\sigma_r\}=\pm 1} e^{-H/kT}$$

$$= \langle Y\,(X - \langle X\rangle)\rangle \quad . \tag{3.2}$$

If the fluctuations in X are large enough so that the right-hand side of Eq. (3.2) might be more strongly divergent at the critical point than $\langle Y\rangle$, then we say that the operator X is *thermodynamically significant*. Since derivatives with respect to temperature and magnetic field increase the rate of divergence of various physical quantities, the Hamiltonian and total magnetization are, in these terms, thermodynamically significant.

The scaling concept permits a more precise definition of the size of fluctuations in various operators[4] and hence a more precise statement of the concept of thermodynamic significance. According to scaling, when the length scale changes according to

$$\vec{r} \to \vec{r}/L \quad , \tag{3.3a}$$

basic intensive operators scale according to

$$x_\alpha(\vec{r}) \to L^{\nu_\alpha} x_\alpha(\vec{r}/L) \quad , \tag{3.3b}$$

whereas the extensive operators obey

$$X_\alpha = \int d\vec{r}\; x_\alpha(r) \to L^{-(d-\nu_\alpha)} X_\alpha \quad . \tag{3.3c}$$

Here d is the dimensionality of the system and the factor L^{-d} comes from the transformation of $d\vec{r}$. In particular, if $x_\alpha(\vec{r})$ scales as ν_α and $x_\beta(\vec{r})$ scales as ν_β, then their correlation function at the critical point must have an r dependence of the form

$$\langle x_\alpha(\vec{r})\, x_\beta(\vec{r}')\rangle \approx \frac{1}{|\vec{r}-\vec{r}'|^{\nu_\alpha+\nu_\beta}} A_{\alpha\beta} \quad , \tag{3.4}$$

in order that the invariance property under the transformation (3.3a) and (3.3b) may hold. [In this equation $A_{\alpha\beta}$ may depend upon the angular orientation of $(\vec{r}-\vec{r}')$ if x_α and x_β are spinor, vector, or tensor variables.]

The general rule that arises from this form of scaling is that operators which scale as low powers of L can have large fluctuations whereas higher powers of L indicate smaller fluctuations. In particular, negative powers of L on the right-hand side of (3.3b) or (3.3c) indicate the possibility for infinite fluctuations in question whereas positive powers denote bounded fluctuations. Since both σ_r and μ_r are bounded, no intensive operator has infinite fluctuations, hence $\nu_\alpha > 0$. On the other hand, if infinite fluctuations in X_α are to be possible,

$$d - \nu_\alpha \geq 0 \quad . \tag{3.5}$$

This last statement is then the condition for thermodynamic significance.

By definition, thermodynamically significant operators produce infinities at $T = T_c$. Physical arguments are then available to discuss these infinities. For this reason, we shall confine our attention almost completely to thermodynamically significant operators.

B. Order and Disorder Variables

The variables μ_r and σ_r are each thermodynamically significant. Since at the critical point μ and σ each have the same autocorrelation function, their scaling indexes are identical. Their scaling index, which we shall call $\nu_{1/2}$, is

$$\nu_\sigma = \nu_\mu \equiv \nu_{1/2} = \tfrac{1}{8} \quad . \tag{3.6}$$

But the σ's and μ's also give other thermodynamically significant operators, since these and the other operators of interest have the property that their correlation functions for large spacial separations are slowly varying in space. Hence, if $\vec{l}$ is any vector which goes from one lattice site to another nearby site, we know that a correlation function

$$\langle[\sigma(\vec{r}+\vec{l}) - \sigma(\vec{r})]x(\vec{r}')\rangle$$

is approximately equal to a dot product of the form $\vec{l}\cdot\vec{V}$ when $|\vec{r}-\vec{r}'| \gg l$. The coefficient $\vec{V}$ is identified with a new correlation function of the form $\langle\vec{x}(\vec{r})x(\vec{r}')\rangle$. It is natural to give to $\vec{x}(\vec{r})$ the name $\vec{\nabla}\sigma(\vec{r})$. From the identification of the gradient operation it follows at once that $\vec{\nabla}x_\alpha(\vec{r})$ has the scaling index $1+\nu_\alpha$. In particular the variables $\nabla\mu_r$ and $\nabla\sigma_r$ have $\nu = 1 + \frac{1}{8}$ and are also thermodynamically significant.

C. Energy Density

The intensive variable conjugate to $\mathcal{E} \sim (T - T_c)/T_c$ is an energy density. We choose as our definition of energy density

$$\mathcal{E}(j+\tfrac{1}{2}, k+\tfrac{1}{2}) = [\sigma_{jk}\sigma_{j+1,k} - \tanh 2K_x(j+\tfrac{1}{2}, k)]$$
$$\times \tanh 2K_x(j+\tfrac{1}{2}, k)$$
$$+ [\sigma_{jk}\sigma_{j,k+1} - \tanh 2K_y(j, k+\tfrac{1}{2})]$$
$$\times \tanh 2K_y(j, k+\tfrac{1}{2}) \quad . \tag{3.7}$$

This particular form is taken to produce a simple transformation law for $\mathcal{E}$ under the K-W transform. Since

$$\langle\sigma_{jk}\sigma_{j+1,k}\rangle\{K\} = \frac{\partial}{\partial K_x(j+\frac{1}{2}, k)} \ln Z\{K\} \quad ,$$

it follows from Eq. (3.7) that

$$\langle \mathcal{E}(j+\tfrac{1}{2}, k+\tfrac{1}{2})\rangle \{K\} = \left(\tanh 2K_x(j+\tfrac{1}{2}, k)\frac{\partial}{\partial K_x(j+\tfrac{1}{2}, k)} + \tanh 2K_y(j, k+\tfrac{1}{2})\frac{\partial}{\partial K_y(j, k+\tfrac{1}{2})}\right)\ln Y\{K\} \quad .$$

Application of the transform then indicates that

$$\langle \mathcal{E}(r)\rangle\{K\} \equiv -\langle \mathcal{E}(r^*)\rangle\{K^*\} \quad ,$$

so that under the K-W transformation

$$\mathcal{E}(r) \rightarrow -\mathcal{E}(r^*) \quad .$$

From the known[13] energy-energy correlation function and Eq. (3.4) it follows that $\mathcal{E}$ scales as $1/r$ and also that the gradient of $\mathcal{E}$ is also thermodynamically significant.

D. Stress Tensor

From the definition of $\mathcal{E}(\vec{r})$, the integral over all space of the energy density is an operator which has, as its main effect, a derivative with respect to temperature, or more properly a derivative with respect to $\mathcal{E}$ at fixed u. But, we also need an intensive variable whose extensive partner varies u, the asymmetry parameter.

Near the critical point, all correlation functions depend upon a distance variable with a kind of elliptical symmetry[14]:

$$\mathcal{R}^2 = x^2/u^2 + u^2y^2 \tag{3.8}$$

in the limit of $\mathcal{R}$ much larger than a lattice constant. Therefore, a derivative with respect to u is equivalent to straining the system according to

$$u\frac{\partial}{\partial u} = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad . \tag{3.9}$$

There is an extensive operator which quite precisely performs the differentiation operation indicated in Eq. (2.29). Its intensive partner is

$$t_1(j+\tfrac{1}{2}, k+\tfrac{1}{2}) = \tfrac{1}{2}\tanh 2K_x(j+\tfrac{1}{2}, k)[\sigma_{jk}\sigma_{j+1,k} - \tanh 2K_x(j+\tfrac{1}{2}, k)] - \tfrac{1}{2}\tanh 2K_y(j, k+\tfrac{1}{2})[\sigma_{jk}\sigma_{j,k+1} - \tanh 2K_y(j, k+\tfrac{1}{2})] \quad , \tag{3.10}$$

so that

$$\langle t_1(j+\tfrac{1}{2}, k+\tfrac{1}{2})\rangle\{K\} = \tfrac{1}{2}\left[\tanh 2K_x(j+\tfrac{1}{2}, k)\frac{\partial}{\partial K_x(j+\tfrac{1}{2}, k)} - \tanh 2K_y(j, k+\tfrac{1}{2})\frac{\partial}{\partial K_y(j, k+\tfrac{1}{2})}\right]\ln Y\{K\} \quad . \tag{3.11}$$

Notice that t_1 is even under the K-W transform and odd under the transform which replaces x by y. Note that the expression for t_1 is the same as the expression for $\mathcal{E}$ except for a sign change which produces the opposite parity under the K-W transform.

The effect of the extensive operator

$$T_1 = \sum_{j,k} t_1(j+\tfrac{1}{2}, k+\tfrac{1}{2}) \tag{3.12}$$

is given by Eq. (3.9). In particular, whenever the distance between r_1 and r_2 is large,

$$\langle T_1\sigma_{r_1}\sigma_{r_2}\rangle = \left(-j_1\frac{\partial}{\partial j_1} + k_1\frac{\partial}{\partial k_1} - j_2\frac{\partial}{\partial j_2} + k_2\frac{\partial}{\partial k_2}\right)\langle\sigma_{r_1}\sigma_{r_2}\rangle = \left(\frac{-2(j_1-j_2)^2}{u^2} + 2(k_1-k_2)^2u^2\right)\frac{\partial}{\partial \mathcal{R}_{12}^2}\langle\sigma_{r_1}\sigma_{r_2}\rangle \quad ,$$

where

$$\mathcal{R}_{12}^2 = (j_1-j_2)^2/u^2 + (k_1-k_2)^2u^2$$

is the distance variable which appears in the correlation functions.

According to Eq. (3.9), the effect of T_1 is that of a second-rank tensor. Since the two-dimensional Ising model has, near its critical point, an underlying rotational symmetry, the other component of this second-rank tensor

$$T_2 \rightarrow -\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right) \tag{3.13}$$

must also exist together with an associated density $t_2(j, k)$.

A plausible but so far unchecked expression for $\langle t_2\rangle$ is

$$\langle t_2(j+\tfrac{1}{2}, k)\rangle = -\tfrac{1}{2}\left(\tanh 2K_x(j+\tfrac{1}{2}, k)\tanh 2K_y(j+1, k+\tfrac{1}{2})\frac{\partial}{\partial K_x(j+\tfrac{1}{2}, k)}\frac{\partial}{\partial K_y(j+1, k+\tfrac{1}{2})} - \tanh 2K_x(j+\tfrac{1}{2}, k)\tanh 2K_y(j+1, k-\tfrac{1}{2})\frac{\partial}{\partial K_x(j+\tfrac{1}{2}, k)}\frac{\partial}{\partial K_y(j+1, k-\tfrac{1}{2})}\right)\ln Y\{K\} \quad . \tag{3.14}$$

This identification agrees with Eq. (3.13), in the sense that t_2 is even under the interchange of x and y. The main characteristic of Eq. (3.14) is the presence of the correlations $\langle \sigma_{jk}\sigma_{j+1,k+1}\rangle$ and $\langle \sigma_{jk}\sigma_{j+1,k-1}\rangle$.

In the present work we will not need to use Eq. (3.14) but rather the symmetry properties of t_2, which are independent of its explicit form. Both t_1 and t_2 must be even under the K-W transformation. The intensive variables t_1 and t_2 form a second-order traceless tensor:

$$t_{ij}=\begin{pmatrix} t_1 & t_2 \\ t_2 & -t_1 \end{pmatrix} . \tag{3.15}$$

The scaling properties of t_{ij} are very simple. According to Eqs. (3.9) and (3.13), T_1 and T_2 are invariant under length transforms. Therefore,

$$d-\nu_t=0 ,$$

so that the index for t_{ij} is $\nu_t=2$. Since t_{ij} is itself at the very edge of thermodynamic significance, ∇t_{ij} is not significant.

E. Spinor Variables

The remaining fundamental fluctuating variables needed in the discussion of the two-dimensional Ising model are the variables employed by Onsager, by Kaufman, and later by many other authors. The original Onsager variables are composed of a product of a spin and an adjacent disorder variable as in Fig. 5(a). We call these spinor variables because near T_c correlation functions formed from products of these variables have the rotational properties associated with spinor correlations.

According to the reduction hypothesis, any product of nearby local variables can be expressed as a linear combination of fundamental variables. Consequently, we might expect the product of a μ and a nearby σ to be a linear combination of two fundamental spinor components. The relative placement of the μ and σ will determine the coefficients in the expansion. When $K_x=K_y$, we choose to interpret a product of σ and μ to be analogous to the spinor which has angular momentum directed along the line pointing from σ to μ. Hence, the products shown in Figs. 5(a) are proportional to spinors pointing in directions differing by $\pm 45°$ from the y direction. If a_+ and a_- represent, respectively, spinors with angular momentum in the $+y$ and $-y$ directions, then the two Onsager variables are related to a_+ and a_- by

$$b_+=a_+\cos\tfrac{1}{8}\pi+a_-\sin\tfrac{1}{8}\pi ,$$

$$b_-=i\,(a_+\sin\tfrac{1}{8}\pi+a_-\cos\tfrac{1}{8}\pi) .$$

The two spinor components a_+ and a_-, which we take to be fundamental, are defined by correlations with orientations as shown in Fig. 5(b).

To check that a_+ and a_- have some of the rotational properties of spinors, we study the correlation function composed of one a_+ or a_-, a μ, and a σ as shown in Fig. 6(a). Now rotate the orientation of the a_+ and the a_- clockwise through 180°. If we never let the paths cross in the course of the rotation, the result of the rotation is shown in Fig. 6(b). To bring the paths back to the form shown in Fig. 6(a), we must let the μ paths on the left-hand figure cross through the σ, thereby picking up a minus sign. We find then that a 180° rotation produces $a_+\to -a_-$ and $a_-\to a_+$. Two 180° rotations would then have the effect $a_+\to -a_+$ and $a_-\to -a_-$. These are precisely the correct transformation properties for spinors. The spinor variables have a correlation function which was called g in Ref. 9. Hecht has evaluated the asymptotic form of this correlation at $T\to T_c$ and finds an inverse first-power dependence upon the separation distance [see Hecht[15] Eq. (4.50)]. Then from Eq. (3.4) it follows that the spinor variables scale as $r^{-1/2}$. Therefore, their gradients are thermodynamically significant.

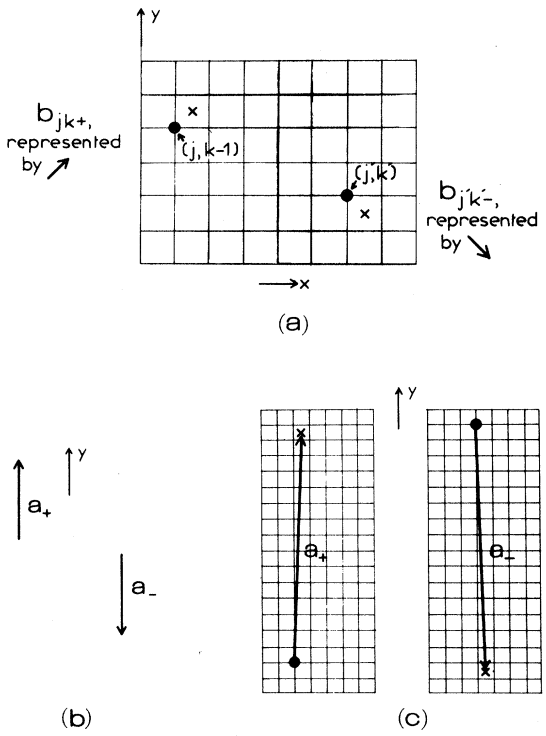


FIG. 5. In the case $K_x=K_y$ the spinors b_+ and b_- admit a graphic representation indicated in (a). As before, a cross (×) represents a μ variable, a dot (•) a σ variable. The spinors a_+ and a_- are shown in (b). This figure needs to be interpreted as the limiting case of (c) when the y distance between the σ and μ goes to infinity. Alternatively, they can be interpreted as the limiting case in which the interaction along the y axis is vanishingly small.

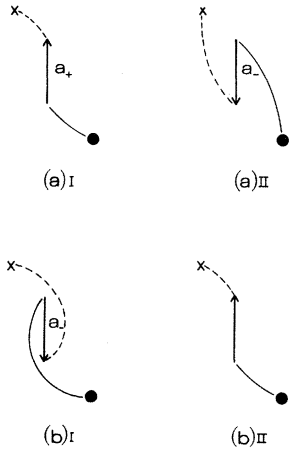


FIG. 6. We draw a spinor variable, paths, and a σ and μ. The lattice has been omitted and the paths are drawn as curves. The operations indicated sustain the interpretation of a_+ and a_- as having spinor character. (i) Rotating a_+ by 180° clockwise, we go from (a)I to (b)I. To go from (b)I to (a)II the μ path needs to cross the "tail of the arrow," i.e., the σ contained in a_-. Hence $a_+ \to -a_-$ under the clockwise rotation. (ii) Rotating a_- by 180° clockwise in (a)II we end up in (b)II, which is identical to (a)I. Therefore $a_- \to a_+$ under this rotation. By repeating this process twice we find that a 360° rotation produces $a_+ \to -a_+$; $a_- \to -a_-$, the well known rotation property of spinors.

IV. CORRELATIONS ALONG A LINE

A. Basic Result

In Appendix B, the Onsager theory is employed to evaluate a $T = T_c$ correlation function composed of μ's and σ's being on a single line and separated by many lattice constants. The type of correlation function is shown in Fig. 7. Notice that the μ's lie slightly to the right of the y axis, and their paths all lie to the right of the σ's.[16] To write this result, we define

$$D_{1/2}(r) = \sigma_r \,, \quad D_{-1/2}(r) = \mu_r \,, \tag{4.1}$$

and consider an average of a product of operators $D_{\gamma_i}(r_i)$

$$\left\langle \prod_{i=1}^{N} D_{\gamma_i}(r_i) \right\rangle = \langle X \rangle \,, \tag{4.2}$$

arranged on the y axis so that $y_i < y_{i+1}$. This average is evaluated in Appendix B for the special case $T = T_c$. The key part of this result is the statement that there is a number Γ describing the product in (4.2). This "quantum number" is defined recursively as

$$\Gamma_1 = \gamma_1 \,, \quad \Gamma_{i+1} = \Gamma_i + (-1)^{2\Gamma_i}\gamma_{i+1} \,, \tag{4.3}$$

and $\Gamma \equiv \Gamma_N$. Our basic result can be stated as

$$\langle X \rangle = 0 \quad \text{for all } \Gamma \neq 0 \,. \tag{4.4}$$

It is important to understand why the Γ defined in these terms determines the vanishing or nonvanishing of the correlation.

B. "Quantum Number" Γ

It is easy to understand why correlations vanish for half-integral values of Γ, for if Γ is half-integral, then the correlation function is necessarily a product of an odd number of terms. Therefore it must contain either an odd number of μ's[17] or an odd number of σ's. But every critical-point correlation function must be even under the spin-flip operation

$$\sigma_r \to -\sigma_r \quad \text{for all } r \tag{4.5}$$

and by the K-W symmetry also even under the μ flip

$$\mu_r \to -\mu_r \quad \text{for all } r \,. \tag{4.6}$$

But all correlation functions with odd N are odd under the simultaneous application of (4.5) and (4.6). Since they must be both odd and even, they are zero.

The vanishing of the correlations when Γ is an odd integer is also easy to understand. In this case, the product contains an odd number of both σ's and μ's, so that either the symmetry (4.5) or the symmetry (4.6) is sufficient to ensure a vanishing average.

But, the case in which Γ is an even integer is much harder. Why do the correlation functions shown in Figs. 7(d) and 7(e) behave so differently? They contain the same operators, just ordered differently upon the line. The first of these has $\Gamma = 0$, and is certainly nonvanishing; the second has $\Gamma = 2$ and vanishes. Why?

A partial understanding of this behavior can be obtained if we remember that when r_1 and r_2 are close together and $y_1 < y_2$, then

$$\sigma_{r_1}\mu_{r_2} \sim a_+ \,, \tag{4.7a}$$

$$\mu_{r_1}\sigma_{r_2} \sim a_- \,. \tag{4.7b}$$

If we were working in three-dimensional space, we would agree that a product of operators of the form (4.7a) adds $\frac{1}{2}$ unit of J_y whereas the combination (4.7b) adds $-\frac{1}{2}$ unit of J_y, where J_y is the angular momentum in the y direction. Then the correlation function in Fig. 7(e) would have $J_y = 1$ whereas the correlation function in Fig. 7(d) would have $J_y = 0$. Therefore, in three-dimensional space, the correlation function Fig. 7(d) would be nonvanishing and the one in Fig. 7(e) would vanish.

This angular momentum argument gives the right answer for all even values of N. To find J_y, we pair adjacent operators in the product. We ascribe

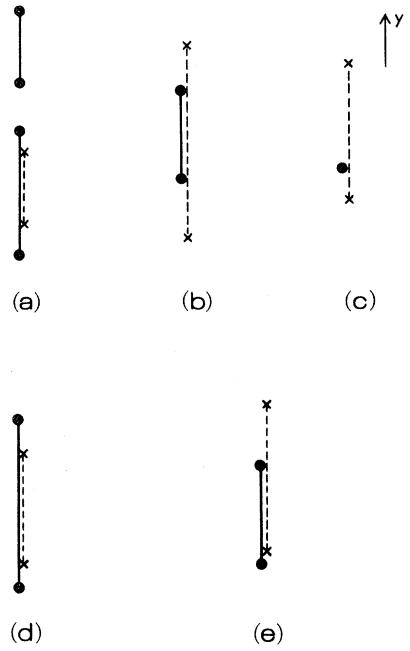


FIG. 7. Different cases of correlation functions along a line. The μ paths lie slightly to the right of the σ paths. The γ values for the different cases shown are $\gamma_a=\gamma_b=\gamma_d=0$, $\gamma_c=-\frac{3}{2}$, $\gamma_e=2$. The correlation functions of the operators in (c) and (e) vanish. Note that (e) has the same operators as in (b) and (d) but in a different arrangement.

$J_y=+\frac{1}{2}$ to all products of the type (4.7a), $J_y=-\frac{1}{2}$ to all products of the type (4.7b), and $J_y=0$ to products $\mu_{r_1}\mu_{r_2}$ and $\sigma_{r_1}\sigma_{r_2}$. Then the total J_y is the same as $\frac{1}{2}\Gamma$. The average is nonvanishing if and only if J_y is zero.

Since we are not working in three dimensions, this angular momentum analogy is far from perfect. Nonetheless, it is clear that the quantum number Γ, which describes the relative ordering of σ's and μ's, describes an absolutely essential quality of the operators on a line and that this quality bears some relation to angular momentum.

From Eq. (4.3) we can work out a composition rule: Given two operators $X_\alpha(r_1)$ and $X_\beta(r_2)$ with quantum numbers α and β, such that all spins and μ's in the X_α lie below, i.e., have smaller y coordinate, all spins and μ's in X_β, then the product of these two is another operator with well-defined Γ,

$$X_\alpha(r_1)X_\beta(r_2)=X_\gamma\,,\qquad \gamma=\alpha+(-1)^{2\alpha}\beta\,. \tag{4.8}$$

Consequently, the Γ values add if the first operator has integral γ, and subtract if the first has half-integer γ.

The Γ quantum number will play an essential role in our further discussions. Its important properties are: (a) the composition rules (4.3) and (4.8); (b) the fact that nonvanishing operators with different Γ's behave differently near the critical point; (c) only operator products with total Γ equal to zero have nonvanishing averages at $T=T_c$.

C. Higher-Order Operators

The other key result for our analysis is the actual value of the average (4.2)

$$\left\langle \prod_{i=1}^{N} D_{\gamma_i}(r_i)\right\rangle = \begin{cases} 0 & \text{if } \Gamma\neq 0 \\ \displaystyle\prod_{1\le i<j\le N}[f(i,j)]^{\gamma_i\gamma_j p_i p_j} & \text{if } \Gamma=0\,. \end{cases} \tag{4.9}$$

Here

$$p_i=(-1)^{2\Gamma_i}\,,$$
$$f(i,j)=\langle\sigma(r_i)\sigma(r_j)\rangle^{-4}\approx|r_i-r_j|\,c\,, \tag{4.10}$$

where c is a known constant. The results (4.9) and (4.10) hold at $T=T_c$ whenever $|r_i-r_j|$ is much greater than a lattice constant and all the r_i lie on a single straight line. (We neglect the fact that the μ's are not exactly on the same line than the σ's because this is unimportant here. But see Sec. IV D.)

The result (4.9) describes correlations of operators D_γ with γ value equal to $\pm\frac{1}{2}$. The reduction idea permits us to use (4.9) to define and determine the properties of a much larger set of operators: the set $D_\gamma(r)$ with γ being any positive or negative integer or half-integer. The basic idea is that the higher-order operator is formed from a set of σ's and μ's closely spaced upon the line. For convenience we allow the σ's and μ's to alternate and the spacing between the neighboring operators to be the small distance a, as in Fig. 8. We then define, for positive γ,

$$X_\gamma(\vec{R})=\sigma(\vec{r}_1)\times\mu(\vec{r}_2)\times\sigma(\vec{r}_3)\times\mu(\vec{r}_4)\cdots\ (2\gamma\text{ terms})\,, \tag{4.11a}$$

with

$$y_{i+1}=y_i+a,\quad x_i=0\,,$$
$$\vec{R}=\frac{1}{2\gamma}\sum_{i=1}^{2\gamma}\vec{r}_i\,.$$

The same definition holds for negative γ, except that each σ is replaced by a μ, and vice versa,

$$X_\gamma(\vec{R})=\mu(\vec{r}_1)\times\sigma(\vec{r}_2)\times\mu(\vec{r}_3)\cdots\ (2|\gamma|\text{ terms})\,. \tag{4.11b}$$

For completeness, we also define

$$X_0(\vec{R})=1\,. \tag{4.11c}$$

From the reduction idea, we might expect that as $a\to 0$, the $X_\gamma(\vec{R})$ becomes a function of a times an operator independent of a. In symbols

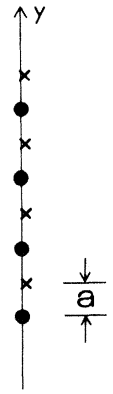


FIG. 8. Operators with $|\gamma| > \frac{1}{2}$ are constructed with the use of the two types of fundamental entities, σ's and μ's spaced along the line. In this figure, $\gamma = 4$.

$$X_\gamma(\vec{R}) \to A_\gamma(a) D_\gamma(\vec{R}) , \tag{4.12}$$

where A_γ is a set of coefficients dependent upon a and γ, and $D_\gamma(\vec{R})$ is an operator.

Equation (4.9) can then be extended to include correlations among the higher-order D_γ. Let r_1, $r_2, \ldots, r_{2\gamma}$ be spaced as in Eq. (4.11a) and let

$$|r_k - r_{2|\gamma|}| \gg a \quad \text{for } 2|\gamma| < k \leq N .$$

We require also that for $i \leq 2|\gamma|$ the operators alternate as in Fig. 8. Then from Eq. (4.9) the resulting correlation function takes the form

$$\left\langle X_\gamma(\vec{R}) \prod_{k=2|\gamma|+1}^{N} D_{\gamma_k}(r_k) \right\rangle = \begin{cases} 0 & \text{if } \Gamma \neq 0 \\ ABC & \text{if } \Gamma = 0 \end{cases} ,$$

where

$$A = \prod_{1 \leq i < j \leq 2|\gamma|} [f(i,j)]^{1/4} ,$$

$$B = \prod_{1 \leq i \leq 2|\gamma|} \prod_{2|\gamma| < k \leq N} [f(i,k)]^{\gamma_i \gamma_k p_i p_k} , \tag{4.13}$$

$$C = \prod_{2|\gamma| < k < m \leq N} [f(k,m)]^{\gamma_k \gamma_m p_k p_m} .$$

Notice that C does not depend upon a. As $a \to 0$ all the r_i in B approach one another and B is a product of $2|\gamma|$ identical terms. Hence,

$$B \to \prod_{2|\gamma| < k \leq N} [f(R - r_k)]^{\gamma \gamma_k p p_k} \left[1 + O\left(\frac{a^2}{(R - r_k)^2}\right)\right] . \tag{4.14}$$

The correction term becomes negligible as $a \to 0$. In Eq. (4.14) the p is equal to the p value of the operator with the smallest y coordinate in X_γ as defined in Eq. (4.9). So, in this case it is $p = -1$.

Finally, the A defined by Eq. (4.13) is a number which depends only upon γ and a but not upon the values of r_k and γ_k for $k \geq 2|\gamma|$. Hence, it is possible to identify the A in Eq. (4.13) with the quantity $A_\gamma(a)$ defined by Eq. (4.12). Therefore, when Eq. (4.12) is substituted in Eq. (4.13), the A's cancel out and we find that

$$\left\langle D_\gamma(R) \prod_{k=2|\gamma|+1}^{N} D_{\gamma_k}(r_k) \right\rangle = \begin{cases} 0 & \text{for } \Gamma \neq 0 \\ BC & \text{for } \Gamma = 0 \end{cases} . \tag{4.15}$$

Furthermore, a term by term comparison of Eq. (4.15) with Eq. (4.9) indicates that these equations have precisely the same structure, except that the first γ has been extended to have possible values $\pm 1, \pm \frac{3}{2}, \pm 2, \ldots$, also. Therefore, successive applications of the same logic enable us to extend our arguments and prove that Eq. (4.9) and the associated Eq. (4.3) hold for operators D_{γ_k} with all possible values of γ_k.

The only case not discussed so far is $\gamma_k = 0$. But if we define

$$D_0(r) = 1 , \tag{4.16}$$

then this case is covered also.

D. Symmetry Properties

The operators D_γ have particularly simple symmetry properties under the symmetry transformations of the Ising model. For example, under $y \to -y$, the X_γ go into themselves for γ half-integral and into $X_{-\gamma}$ for γ integral (see Fig. 9). Hence, the D_γ's obey

$$y \to -y ,$$

which implies

$$D_\gamma(y) \to D_{\gamma'}(-y), \quad \gamma' = -(-1)^{2\gamma}\gamma . \tag{4.17a}$$

The transformation $x \to -x$ is a bit more subtle. At first sight, it would appear that since all operators are defined by the placement of operators on the y axis the transformation $x \to -x$ cannot possibly make any difference, i.e., under $x \to -x$, $D_\gamma \to D_\gamma$. But this statement neglects the fact that the D_γ are defined with the μ_r appearing infinites-

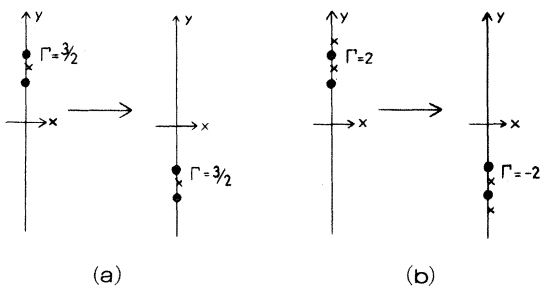


FIG. 9. Upon the transformation $y \to -y$ the operators X_γ, with γ being a half-integral, go into themselves (a), and the ones with γ integral into $X_{-\gamma}$ (b).

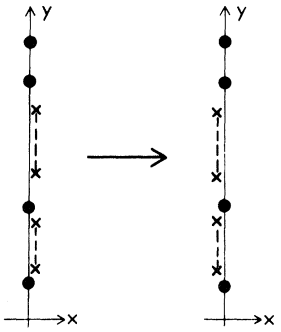


FIG. 10. Due to the fact that the μ's are not exactly on the y axis, the transformation $x \to -x$ is not the identity. The restoration of the operators to their "canonical" arrangement can introduce a minus sign in X_γ, related with the number of σ variables crossed by the μ paths in this process.

imally to the right, say, of the y axis. Hence, under $x \to -x$ the μ paths are displaced to the left of the y axis (see Fig. 10). Extra minus signs can appear as the paths are returned to their "canonical" position, just to the right of the y axis. Hence, we conclude that

$$x \to -x$$

implies

$$D_\gamma(r) \to D_\gamma(r)\Lambda_\gamma(r) , \tag{4.17b}$$

where $\Lambda_\gamma(r)$ is ± 1, depending upon whether or not the path displacement gives a net change of sign.

A similar problem complicates the discussion of the K-W transformation. If path complications are neglected, the K-W transform simply replaces all μ_r by σ_r, and vice versa. Therefore, in the absence of these complications the transform of D_γ is $D_{-\gamma}$. But as the σ's and μ's are interchanged, the μ's end up to the left of the y axis. The displacement of the μ's to their canonical position once again gives minus signs, so that K-W transform implies

$$D_\gamma(r) \to D_{-\gamma}(r)\Lambda_\gamma(r) . \tag{4.18}$$

Since the "parity" factors $\Lambda_\gamma(r)$ are the same in (4.17b) and (4.18), both transformations together give a result which does not contain these factors. We call this combined transformation CP and note that

$$CP = (\text{K-W transform}) \times (x \to -x)$$

implies

$$D_\gamma(r) \to D_{-\gamma}(r) . \tag{4.19}$$

Equation (4.19) evades the question of the parity of the operators D_γ under $x \to -x$. But the value of $\Lambda_\gamma(r)$ is important to our physical identification of the operators D_γ in terms of the already mentioned $a_\pm$, σ, μ, ϵ, and t_{ij}. Therefore, we must write down the values of $\Lambda_\gamma(r)$. By definition $\Lambda_\gamma(r) = \pm 1$, depending upon whether the translation of the μ paths across the y axis requires the μ paths to cut through the σ's in D_γ an even or odd number of times. The evaluation of $\Lambda_\gamma(r)$ requires only a careful enumeration of the possible cases.

For $\gamma = 0$ or $-\frac{1}{2}$ there are no σ's to be considered, and Λ is 1. More generally, for integral n if

$$\gamma = 4n$$

or

$$\gamma = 4n - \tfrac{1}{2} ,$$

then

$$\Lambda_\gamma = 1 . \tag{4.20a}$$

Consider for instance the case $\gamma = 3\frac{1}{2}$. This corresponds to an operator with four σ's and three μ's ordered as in Eq. (4.11a). There are two possible forms of drawing the μ paths, depending on the relative location of D_γ on the line [see Figs. 11(a) and 11(b)]. The first μ in D_γ (i.e., the one with smallest y coordinate) is either connected to another μ variable "below" it, with the second and third interconnected by a path, or the first one is connected to the second, in which case the last need to be connected to another μ not belonging to D_γ, with bigger y. In both cases, there are two μ paths, each of which is bound to cut through a σ variable. As each cut gives a minus sign, the net result is $\Lambda_\gamma(r) = 1$, as given in Eq. (4.20a).

On the other hand, σ_r and $D_1 \sim \sigma_{r_1}\mu_{r_2}$ each flip sign if there is a μ path going below the position of these operators and remain unchanged if there is no such path. Therefore, if

$$\gamma = 4n + \tfrac{1}{2}$$

or

$$\gamma = 4n + 1,$$

then

$$\Lambda_\gamma = \Lambda_\sigma , \tag{4.20b}$$

$$\Lambda_\sigma(r) = \begin{cases} -1 & \text{if there is a } \mu \text{ path going below } r \\ 1 & \text{otherwise} . \end{cases}$$

This factor $\Lambda_\sigma(r)$ appears in all Λ_γ in which γ describes an operator with odd numbers of σ's in it. Thus, if

$$\gamma = 4n - 1$$

or

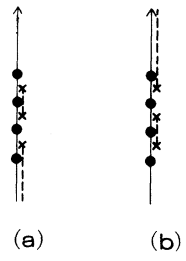


FIG. 11. The figure illustrates the discussion about the determination of the parity Λ_γ. With $\gamma = 3\frac{1}{2}$ there are two possible forms of drawing the μ paths, depending upon the relative placement of $X_{3/2}$ along the line. In both cases the parity is $\Lambda_{3/2} = +1$.

$$\gamma = 4n - \tfrac{3}{2} ,$$

then

$$\Lambda_\gamma(r) = -\Lambda_\sigma(r) , \tag{4.20c}$$

and if

$$\gamma = 4n + 2$$

or

$$\gamma = 4n + \tfrac{3}{2} ,$$

then

$$\Lambda_\gamma(r) = -1 . \tag{4.20d}$$

We can summarize our results by saying that the total parity $\Lambda_\gamma(r)$ is a product of a kind of "intrinsic parity" Λ_γ^i and an "orbital parity" $\Lambda_\gamma^0(r)$ such that

$$\Lambda_\gamma(r) = \Lambda_\gamma^i \Lambda_\gamma^0(r) . \tag{4.21}$$

The orbital parity is just the parity of an individual σ raised to the power of the number of σ's in D_γ, while the intrinsic parity is

$$\Lambda_\gamma^i = \begin{cases} 1 \text{ for } \gamma = 4n - \frac{1}{2},\ 4n,\ 4n + \frac{1}{2},\ 4n+1 \\ -1 \text{ for } \gamma = 4n-2,\ 4n - \frac{3}{2},\ 4n-1,\ 4n + \frac{3}{2} . \end{cases} \tag{4.22}$$

E. Physical Identifications

We have already agreed that $D_0 = 1$, $D_{1/2} = \sigma$, $D_{-1/2} = \mu$, $D_1 = a_+$, and $D_{-1} = a_-$. In this section we try to give plausible physical arguments for the identifications[16]

$$D_{3/2} = +\lambda\, \partial\mu/\partial x, \quad D_{-3/2} = -\lambda\, \partial\sigma/\partial x , \tag{4.23}$$

$$D_{\pm 2} = \pm v t_{12} + w \frac{\partial \mathcal{E}}{\partial x} . \tag{4.24}$$

First let us consider $D_{\pm 2}$. These operators must be even under each of the separate operations

$$\mu_r \to -\mu_r, \quad \sigma_r \to -\sigma_r .$$

Hence, if $D_{\pm 2}$ is to be identified in terms of our previously defined basic operators, it must be proportional to $\mathcal{E}$, t_{ij}, and/or spacial derivatives of $\mathcal{E}$ and t_{ij}. According to Eq. (4.20d), these D's must be odd under $x \to -x$. A proportionality to $\mathcal{E}$, $\partial\mathcal{E}/\partial y$ or t_{11}, is ruled out since each of these operators is even under $x \to -x$. The simplest possible odd operators are t_{12} and $\partial\mathcal{E}/\partial x$. We then try a linear combination

$$D_{\pm 2} = v_\pm t_{12} + w_\pm \frac{\partial \mathcal{E}}{\partial x} ,$$

where $v_\pm$ and $w_\pm$ are coefficients to be determined. Under $y \to -y$, $D_2 \to D_{-2}$ from Eq. (4.16). Under this operation, $\partial\mathcal{E}/\partial x$ is even and t_{12} is odd. As a result

$$v_\pm = \pm v, \quad w_\pm = w ,$$

so that our symmetry principles naturally lend to Eq. (4.24).

As a check, let us apply the CP transform. Under this transform t_{12} changes sign because it is even under K-W but odd by $x \to -x$. On the other hand, $\partial\mathcal{E}/\partial x$ is odd under both parts of the transform and, hence, even in total. As a result, under CP

$$D_2 = v t_{12} + w \frac{\partial \mathcal{E}}{\partial x} \to -v t_{12} + w \frac{\partial \mathcal{E}}{\partial x} \equiv D_{-2} .$$

This result checks with Eq. (4.19).

As a further check, notice that Eq. (4.9) implies

$$\langle D_2(r_1) D_{-2}(r_2) \rangle = [f(r_1 - r_2)]^{-4} \sim |r_1 - r_2|^{-4} .$$

But $\nabla\mathcal{E}$ and t_2 are each known to scale as r^{-2}. Hence the correlation of two such operators should be expected to scale as r^{-4}.

Next, consider $D_{\pm 3/2}$. We know that $D_{3/2}$ changes sign when μ changes sign but not when σ changes sign since $X_{3/2} = \sigma(r_1)\mu(r_2)\sigma(r_3)$. Hence, $D_{3/2}$ may be expected to be proportional to μ or gradients thereof, whereas $D_{-3/2}$ should be proportional to σ and gradients of σ. According to Eq. (4.20), $D_{\pm 3/2}$ behaves the same way as $(\partial/\partial x) D_{\mp 1/2}$ under $x \to -x$. Hence, it is natural to guess

$$D_{\pm 3/2} = \lambda_\pm \frac{\partial}{\partial x} D_{\mp 1/2} .$$

Then CP invariance implies $\lambda_+ = -\lambda_-$, so that Eq. (4.23) follows. To check this result, notice that $D_{\pm 3/2}$ are each even under $y \to -y$, as required by Eq. (4.16). Note also that

$$\langle \sigma_{r_1} \sigma_{r_2} \rangle \sim |r_1 - r_2|^{-1/4}$$

implies via Eq. (4.23) that

$$\langle D_{-3/2}(r_1) D_{-3/2}(r_2) \rangle \sim |r_1 - r_2|^{-9/4} .$$

But Eq. (4.9) implies

$$\langle D_{-3/2}(r_1) D_{-3/2}(r_2) \rangle = [f(r_1 - r_2)]^{-9/4} ,$$

so that the correlation function has its expected value. This check further confirms our hypothesis (4.23).

V. REDUCTION FORMULAS

In this section we make use of the symmetry properties of the A's to suggest a set of reduction formulas for products of two nearly D_γ on the line $x = 0$. The results thereby obtained are then checked against the known results for multiple correlations of D_γ.

A. Hypothesis

To write down the reduction formulas we consider products of the form

$$D_\alpha(\vec{r}_1) D_\beta(\vec{r}_2) = P , \tag{5.1}$$

with r_1 and r_2 lying on the y axis and $y_1 < y_2$. Since the product (5.1) has "quantum number"

$$\gamma = \alpha + (-1)^{2\alpha}\beta , \tag{5.2}$$

it is reasonable to guess that

$$P = A_{\alpha,\beta}(r) D_\gamma(\vec{\vec{R}}) , \tag{5.3}$$

where $A_{\alpha,\beta}$ is a number, $r = |\vec{r}_1 - \vec{r}_2|$, and $\vec{\vec{R}}$ is a coordinate on the y axis in the neighborhood of $\vec{r}_1$ and $\vec{r}_2$. If we use the reference point

$$\vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) , \tag{5.4}$$

we can assume that $\vec{\vec{R}}$ differs from $\vec{R}$ by only a small amount and rewrite the hypothesis (5.3) as

$$D_\alpha(\vec{r}_1) D_\beta(\vec{r}_2) \approx A_{\alpha,\beta}(r) D_\gamma(\vec{R}) + B_{\alpha,\beta}(r) \frac{\partial}{\partial Y} D_\gamma(\vec{R}) \tag{5.5}$$

for $\gamma = \alpha + (-1)^{2\alpha}\beta \neq 0$.

Equation (5.5) is a reasonable consequence of the hypothesis that $D_\gamma(R)$ is the only fluctuating operator with quantum number equal to γ. This hypothesis appears reasonable for $\gamma \neq 0$. However, it is possible to identify several operators with $\gamma = 0$. Explicit calculations show that for r_1 and r_2 close together on the y axis,

$$\begin{aligned} D_{1/2}(\vec{r}_1) D_{1/2}(\vec{r}_2) &= \sigma_{r_1}\sigma_{r_2} \\ &= A_{1/2,1/2}(r) + \overline{B}_{1/2,1/2}(r)\mathcal{E}(\vec{R}) \\ &\quad + C_{1/2,1/2}(r)\, t_{11}(\vec{R}) . \end{aligned}$$

Hence $D_0(R) = 1$, $\mathcal{E}(\vec{R})$, and $t_{11}(\vec{R})$ all have $\gamma = 0$. Therefore, when the γ defined by Eq. (5.2) vanishes, we replace Eq. (5.5) by

$$D_\alpha(\vec{r}_1) D_\beta(\vec{r}_2) = A_{\alpha,\beta}(r) + \overline{B}_{\alpha,\beta}(r)\mathcal{E}(\vec{R}) + C_{\alpha,\beta}(\vec{R}) t_{11}(\vec{R}) \tag{5.6}$$

when $\gamma = \alpha + (-1)^{2\alpha}\beta = 0$.

B. Symmetry Properties

The coefficients in Eqs. (5.5) and (5.6) are limited by the symmetry properties listed in Sec. IV D. For example, the CP transformation which takes

$$\begin{aligned} &\vec{r}_1 \to \vec{r}_1, \quad \vec{r}_2 \to \vec{r}_2, \quad \vec{\vec{R}} \to \vec{\vec{R}} , \\ &D_\gamma \to D_{-\gamma}, \quad \mathcal{E}(R) \to -\mathcal{E}(R), \quad t_{11}(R) \to t_{11}(R) \end{aligned} \tag{5.7}$$

is an exact symmetry of the Ising model. The application of this symmetry to Eq. (5.6) indicates that

$$D_{-\alpha}(r_1) D_{-\beta}(r_2) = A_{\alpha,\beta}(r) - \overline{B}_{\alpha,\beta}(r)\mathcal{E}(R) + C_{\alpha,\beta}(r) t_{11}(R) \tag{5.8}$$

for $\gamma = \alpha + (-1)^{2\alpha}\beta = 0$. But Eq. (5.6) indicates directly that the right-hand side of (5.8) is

$$A_{-\alpha,-\beta}(r) + \overline{B}_{-\alpha,-\beta}(r)\mathcal{E}(\vec{R}) + C_{-\alpha,-\beta}(r) t_{11}(\vec{R}) .$$

Therefore, C is even under the change in sign of both subscripts and $\overline{B}$ is odd under such an interchange. The general conclusion drawn from this symmetry and Eqs. (5.5) and (5.6) is

$$\begin{aligned} &A_{-\alpha,-\beta} = A_{\alpha,\beta}, \qquad B_{-\alpha,-\beta} = B_{\alpha,\beta} , \\ &\overline{B}_{-\alpha,-\beta} = -\overline{B}_{\alpha,\beta}, \quad C_{-\alpha,-\beta} = C_{\alpha,\beta} . \end{aligned} \tag{5.9}$$

A precisely similar logic may be applied to the transformation $y \to -y$, which takes

$$\begin{aligned} &\vec{r}_1 \to -\vec{r}_1, \quad \vec{r}_2 \to -\vec{r}_2, \quad \vec{\vec{R}} \to -\vec{\vec{R}} , \\ &\alpha \to \alpha' = -(-1)^{2\alpha}\alpha, \quad \beta \to \beta' = -(-1)^{2\beta}\beta , \\ &\gamma \to \gamma' = -(-1)^{2\gamma}\gamma . \end{aligned}$$

After this transformation, Eq. (5.5) reads

$$D_{\beta'}(-r_2) D_{\alpha'}(-r_1) = A_{\alpha,\beta}(r) D_{\gamma'}(-R) + B_{\alpha,\beta}(r) \frac{\partial}{\partial Y} D_{\gamma'}(-R) . \tag{5.10}$$

A direct application of Eq. (5.5) indicates that the right-hand side of (5.10) is also

$$A_{\beta',\alpha'}(r) D_{\gamma'}(-R) + B_{\beta',\alpha'}(r)\left(-\frac{\partial}{\partial Y}\right) D_{\gamma'}(-R) .$$

By this logic we find

$$A_{\beta',\alpha'} = A_{\alpha,\beta}, \qquad B_{\beta',\alpha'} = -B_{\alpha,\beta} , \tag{5.11}$$

with

$$\alpha' = -(-1)^{2\alpha}\alpha , \quad \beta' = -(-1)^{2\beta}\beta .$$

The application of this logic to Eq. (5.6) indicates that (5.6) satisfies this symmetry identically.

C. Check of Reduction Formulas

From Eq. (4.9),

$$\left\langle \prod_{i=1}^{N+1} D_{\gamma_i}(r_i) \right\rangle = \prod_{1 \le i < j \le N+1} [f(|r_i - r_j|)]^{\gamma_i \gamma_j p_i p_j} . \tag{5.12}$$

Now allow r_N and r_{N+1} to approach each other. Write $\gamma_N = \alpha$, $\gamma_{N+1} = \beta$; $\gamma = \alpha + (-1)^{2\alpha}\beta$. Equation (5.12) has a right-hand side which is a product of three terms:

$$\begin{aligned} A &= f(|r_N - r_{N+1}|)^{\alpha\beta(-1)^{2\alpha}} , \\ U &= \prod_{1 \le i \le N-1} [f(|r_i - r_N|)^{\alpha} f(|r_i - r_{N+1}|)^{\beta(-1)^{2\alpha}}]^{\gamma_i p_i p_N} , \\ V &= \prod_{1 \le i \le j \le (N-1)} [f(|r_i - r_j|)]^{\gamma_i p_i \gamma_j p_j} . \end{aligned} \tag{5.13}$$

When $\gamma \neq 0$, U may be simplified to

$$U=\prod_{1\le i\le N-1}[f(|r_i-\bar{R}|)]^{\gamma\gamma_i p_i p_N}\left[1+O\left(\frac{r^2}{|r_i-r_N|^2}\right)\right], \tag{5.14}$$

with

$$\vec{\bar{R}}=[\alpha\vec{r}_N+(-1)^{2\alpha}\beta r_{N+1}]/\gamma\,,\qquad r=r_{N+1}-r_N\,. \tag{5.15}$$

When this simplification is made, the right-hand side of (5.12) is precisely of the form

$$A\times\left\langle\prod_{i=1}^{N-1}D_{\gamma_i}(r_i)D_\gamma(\bar{R})\right\rangle.$$

Consequently, we find that for $\gamma\neq 0$,

$$D_\alpha(r_1)D_\beta(r_2)=A\,D_\gamma(\bar{R})\,, \tag{5.16}$$

as asserted in Eq. (5.3). In this way, we verify the basic correctness of our reduction hypothesis. To calculate A and B, note

$$\vec{\bar{R}}=\vec{R}+\frac{\alpha-(-1)^{2\alpha}\beta}{2[\alpha+(-1)^{2\alpha}\beta]}(\vec{r}_1-\vec{r}_2)\,,$$

so that a comparison of Eqs. (5.13) and (5.5) indicates that $A=A_{\alpha,\beta}$ is

$$A_{\alpha,\beta}(r)=[f(r)]^{\alpha\beta(-1)^{2\alpha}}=[\langle\sigma_r\sigma_0\rangle]^{-4\alpha\beta(-1)^{2\alpha}}\,, \tag{5.17a}$$

whereas

$$B_{\alpha,\beta}(r)=-A_{\alpha,\beta}(r)\frac{\alpha-(-1)^{2\alpha}\beta}{2[\alpha+(-1)^{2\alpha}\beta]}r\,. \tag{5.17b}$$

When $\gamma=0$, this approach does not work since the denominator of B diverges. Instead, we note that in this case, as $r=r_{N+1}-r_N$ goes to zero,

$$U=1-\alpha r p_N\sum_{i=1}^{N-1}\frac{\gamma_i p_i}{|r_i-R|}+\tfrac{1}{2}\alpha^2r^2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{\gamma_i\gamma_j p_i p_j}{|r_i-R|\,|r_j-R|}+O\left(\frac{r^3}{|r_i-R|^3}\right). \tag{5.18}$$

The first term in U is identified with correlations of $D_0(R)=1$, the second term with correlations of $\mathcal{E}(R)$, and the third with correlations of $t_{11}(R)$. We then find for $\gamma=0$, $A_{\alpha,\beta}$ is still given by Eq. (5.17a), whereas

$$\bar{B}_{\alpha,\beta}(r)=-\alpha r b\,A_{\alpha,\beta}(r)\,, \tag{5.19a}$$

$$C_{\alpha,\beta}(r)=\tfrac{1}{2}\alpha^2r^2d\,A_{\alpha,\beta}(r)\,, \tag{5.19b}$$

where b and d are constants. With this identification, the correlations of $\mathcal{E}(R)$ and $t_{11}(R)$ with a group of D_γ's may be computed by writing

$$X=\prod_{i=1}^{N}D_{\gamma_i}(r_i)\,,$$

$$\langle X\mathcal{E}(R)\rangle=\frac{\langle X\rangle}{b}\sum_{i=1}^{N}\frac{\gamma_i p_i q_R}{(r_i-R)}\,, \tag{5.20a}$$

$$\langle X t_{11}(R)\rangle=\frac{\langle X\rangle}{d}\sum_{i,j=1}^{N}\frac{\gamma_i\gamma_j p_i p_j}{|r_i-R|\,|r_j-R|}\,, \tag{5.20b}$$

where $q_R=(-1)^{\Gamma_{\inf}}$, and $\Gamma_{\inf}$ is the Γ value for the D operator being just below the position of $\mathcal{E}(R)$, i.e., q_R is equal to the p defined in (4.9) if the corresponding operator has $\gamma=\pm\frac{1}{2}$. Equations (5.20) describe the correlations of X with $\mathcal{E}(R)$ and $t_{11}(R)$ for the special case in which all points in question r_i as well as R lie well separated upon the y axis.

D. Correlation Functions with Several $\mathcal{E}$'s and t's

The next step is to use the same reduction technique to evaluate correlation functions containing several $\mathcal{E}$'s or t_{11}'s.

As an illustration, consider $\langle X\mathcal{E}(R_1)\mathcal{E}(R_2)\rangle$. We start with Eq. (5.20a) and let two operators D_{γ_i} and $D_{\gamma_{j+1}}$ approach each other. We also require that

$$\gamma=\gamma_j+(-1)^{2\gamma_j}\gamma_{j+1}=0.$$

Hence, by using the same argument as in (5.18), we obtain

$$\langle X\mathcal{E}(R_1)\mathcal{E}(R_2)\rangle=\frac{\langle X\mathcal{E}(R_1)\rangle\langle X\mathcal{E}(R_2)\rangle}{\langle X\rangle}+\frac{\langle X\rangle}{b^2}\frac{1}{(R_2-R_1)^2}\,. \tag{5.21}$$

The last term in the right-hand side of (5.21) originates in the terms with γ_j and γ_{j+1} in the sum in Eq. (5.20a).

We notice that setting $X=1$ in (5.21) then gives (note that at $T=T_c$, $\langle\mathcal{E}\rangle=0$)

$$\langle\mathcal{E}(R_1)\mathcal{E}(R_2)\rangle=\frac{1}{b^2}\frac{1}{(R_2-R_1)^2}\,,$$

which coincides with previous calculations.[13] This can be considered as another indication of the consistency and the power of the method. Exactly the same type of calculation gives

$$\langle X\mathcal{E}(R_1)t_1(R_2)\rangle=\frac{\langle X\mathcal{E}(R_1)\rangle\langle X t_{11}(R_2)\rangle}{\langle X\rangle}+\frac{2}{d}\frac{q_{R_1}q_{R_2}}{(R_2-R_1)^2}\langle X\mathcal{E}(R_2)\rangle\,. \tag{5.22}$$

APPENDIX A

We will prove here[18] Eq. (2.11), i.e.,

$$Y\{K\}=Y\{K^*(K)\}\,, \tag{2.11}$$

where

$$Y\{K\}=Z\{K\}P\{K\}\equiv Z\{K\}\,2^{-\mathfrak{N}/2}\prod_{j,k}\left[\cosh 2K_x(j+\tfrac{1}{2},k)\cosh 2K_y(j,k+\tfrac{1}{2})\right]^{-1/2}$$

and $\mathfrak{N}$ is the number of spins in the lattice. By using $\tanh 2K^*(K)=(\cosh 2K)^{-1}$ in P, we obtain

$$\begin{aligned}P\{K\}&=2^{-\mathfrak{N}/2}\prod_{j,k}\left(\frac{\cosh 2K_y^*(j,k-\frac{1}{2})}{\sinh 2K_y^*(j,k-\frac{1}{2})}\,\frac{\cosh 2K_x^*(j-\frac{1}{2},k)}{\sinh 2K_x^*(j-\frac{1}{2},k)}\right)^{-1/2}\\&=P\{K^*\}\,2^{(\mathfrak{N}^*-\mathfrak{N})/2}\prod_{j,k}\left[\sinh 2K_y^*(j,k-\tfrac{1}{2})\sinh 2K_x^*(j-\tfrac{1}{2},k)\right]^{1/2},\end{aligned}\tag{A1}$$

because $\Pi_{j,k}$ covers the whole lattice. Here $\mathfrak{N}^*$ is the number of sites in the dual lattice.

Next, we proceed to write $Z\{K\}$ in terms of K_x and K_y. Use

$$e^{K\sigma\sigma'}=\cosh K+\sigma\sigma'\sinh K,\qquad \cosh K=(\tfrac{1}{2})^{1/2}(\cosh 2K+1)^{1/2},\qquad \tanh K=e^{-2K^*(K)}$$

to get

$$\begin{aligned}Z\{K\}=\sum_{\{\sigma_{j,k}\}=\pm1}\prod_{j,k}\{&\tfrac{1}{2}(e^{K_y^*(j,k-1/2)}+\sigma_{jk}\sigma_{j+1,k}e^{-K_y^*(j,k-1/2)})\,(e^{K_x^*(j-1/2,k)}+\sigma_{jk}\sigma_{j,k+1}e^{-K_x^*(j-1/2,k)})\\&\times[\sinh 2K_y^*(j,k-\tfrac{1}{2})\sinh 2K_x^*(j-\tfrac{1}{2},k)]^{-1/2}\}.\end{aligned}$$

The last square bracket in the right-hand side of this expression is not a function of σ_{jk}, and can be taken out of the summation. It can be seen to cancel the product appearing in Eq. (A1).

Consequently,

$$\begin{aligned}Y\{K\}&=P\{K\}Z\{K\}\\&=2^{(\mathfrak{N}^*-\mathfrak{N})/2}P\{K^*\}\sum_{\{\sigma_{jk}\}=\pm1}\left\{\prod_{jk}(\tfrac{1}{2})(e^{K_y^*(j,k-1/2)}+\sigma_{jk}\sigma_{j+1,k}e^{-K_y^*(j,k-1/2)})(e^{K_x^*(j-1/2,k)}+\sigma_{jk}\sigma_{j,k+1}e^{-K_x^*(j-1/2,k)})\right\}\\&\equiv 2^{(\mathfrak{N}^*-\mathfrak{N})/2}P\{K^*\}T\{K^*\}.\end{aligned}\tag{A2}$$

The last line of Eq. (A2) serves as definition of $T\{K^*\}$.

But $Y\{K^*\}=P\{K^*\}Z\{K^*\}$; hence we must compare $2^{(\mathfrak{N}^*-\mathfrak{N})/2}T\{K^*\}$ with $Z\{K^*\}$, where

$$\begin{aligned}Z\{K^*\}=\sum_{\{\sigma_{jk}\}\pm1}\prod_{j,k}&e^{K_x^*(j+1/2,k)\sigma_{jk}\sigma_{j+1,k}}\\&\times e^{K_y^*(j,k+1/2)\sigma_{jk}\sigma_{j,k+1}}.\end{aligned}$$

We now argue in a manner similar to the case in which all the coupling constants are equal. By expanding the product in $T\{K^*\}$ we notice that it is possible to classify the resulting terms, considering the number of products $(\sigma\sigma')$ included in each one.

So, there will be one term without any $(\sigma\sigma')$, i.e., the one with all the exponentials with positive sign, plus terms with just one pair $(\sigma\sigma')$, ... up to a term with all the pairs $(\sigma\sigma')$'s, i.e., with all the exponentials with negative sign.

Since each one of these terms is summed over all the $\sigma_{jk}=\pm1$, any term with a σ elevated to an odd power will cancel out.

The ones left actually are not a function of $\sigma_{jk}(\sigma_{jk}^{2n}=1)$. Hence each one of them can be taken out of the sum. All will have a common factor (including the factor $2^{(\mathfrak{N}^*-\mathfrak{N})/2}$)

$$2^{(\mathfrak{N}^*-\mathfrak{N})/2}\left(\prod_{j,k}(\tfrac{1}{2})\right)\sum_{\sigma_{jk}=\pm1}1.$$

The summation gives $2^{\mathfrak{N}}$.

To calculate the product, remember that it comes from the transformation of $Z\{K\}$, each factor $(\frac{1}{2})^{1/2}$ being related to one of the coupling constants, K_x or K_y. Thus, we have

$$\prod_{j,k}(\tfrac{1}{2})=(\tfrac{1}{2})^{s/2},$$

with s being the number of coupling constants.

But it has been indicated[19] that the K-W transform is an *exact* symmetry *only* in the case that the lattice can be extended on a simply connected surface. Moreover, to be completely defined, in the sense that *all* the transformed coupling constants really represent an interaction, the surface needs to be spheroidal. In this case, the following relation holds: $\mathfrak{N}+\mathfrak{N}^*=s+2$.

Hence the factor which multiplies each of the terms in $T\{K\}$ is

cede the elements of the $\tau=-1$ subregion. Finally, inside one subregion the elements are set in order of increasing k. This ordering defines the condition $z'<z$.

Because *a posteriori* we will take the lim $C\to\infty$, the products in Eq. (B17) can be expanded by considering C much bigger than any k, k'. Then, for instance

$$a(z)-a(z')\simeq C \quad \text{if } z\in\{\tilde{\eta}_+\} \text{ and } z'\in\{\tilde{\eta}_-\}.$$

To make further progress, let us group the terms of Eq. (B17) into C terms and no-C terms, i.e., terms which do or do not contain the factor C.

So, for instance, the C terms of the numerator are the ones for which

$$z'<z, \quad z'\in\{\tilde{\eta}_+\}, \text{ and } z\in\{\tilde{\eta}_-\},$$

or

$$z'<z, \quad z'\in\{\tilde{\eta}_-\}, \text{ and } z\in\{\tilde{\eta}_+\}.$$

Next, inside each one of these two types of products, we proceed to group terms with z and z' belonging to the same region $\{j\}$, or to two different regions, $\{i\}$ and $\{j\}$.

Let us call S_j and S_{ij} the products of C terms with z and z' in the region $\{j\}$ or the regions $\{i\}$ and $\{j\}$, respectively. Analogously, let us call T_j and T_{ij} the similar products for the no-C terms.

The result of this process is that

$$\det{}_{\tilde{\eta}}h(z,z')=\lim_{C\to\infty}\left(\prod_j S_j \prod_{i<j} S_{ij}\right)\left(\prod_j T_j \prod_{i<j} T_{ij}\right). \tag{B18}$$

Consider the C part first:

$$\prod_j S_j \prod_{i<j} S_{ij}=\prod_j(-1)^{(N_-^j)^2} C^{-(N_+^j-N_-^j)^2}\prod_{i<j} C^{-2(N_+^i-N_-^i)(N_+^j-N_-^j)}$$

$$=(-1)^{\Sigma_j N_-^j} C^{-[\Sigma_i (N_+^i-N_-^i)]^2}. \tag{B19}$$

But the exponent of C must vanish if the determinant is different from zero, because we need to consider the case $C\to\infty$.

Hence, we require

$$\textstyle\sum_i (N_+^i-N_-^i)=0$$

and, consequently, have

$$\prod_j S_j \prod_{i<j} S_{ij}=(-1)^{N_-},$$

where $N_-=\sum_j N_-^j$.

Equation (4.2) follows, with the identification

$$\textstyle\sum_i (N_+^i-N_-^i)\equiv\Gamma. \tag{B20}$$

All the properties described in Sec. IV B can be derived with the help of Eq. (B20). We note that the difference of $(N_+^i-N_-^i)$ vanishes if the operators at k_i and $k_{i'}$ are two σ's or two μ's, whereas the cases $(\sigma_i\mu_{i'})$ and $(\mu_i\sigma_{i'})$ give $(+1)$ and (-1), respectively. We now sketch the derivation of Eq. (4.9), the other basic result. Going back to Eq. (B18), with the condition $\Gamma=0$,

$$\langle X\rangle=[\det_{\tilde{\eta}}h]^{1/2}=\prod_j T_j^{1/2}\prod_{i<j} T_{ij}^{1/2}, \tag{B21}$$

where

$$T_j^{1/2}\equiv F_{jj'}=\prod_{z,z'\in\{j+\}}^{z'<z}[\pi(k-k')]\prod_{z,z'\in\{j-\}}^{z'<z}[\pi(k-k')]\Big/\prod_{\substack{z\in\{j+\}\\ z'\in\{j-\}}}[\pi(k-k'-\tfrac{1}{2})], \tag{B22a}$$

$$T_{ij}^{1/2}\equiv F_{ii'jj'}=\prod_{k'\in\{i+\};\,k\in\{j+\}}[\pi(k-k')]\Big/\prod_{\substack{k\in\{i+\}\\ k'\in\{j-\}}}\pi(k-k'-\tfrac{1}{2})\left(\prod_{k'\in\{i-\};k\in\{j-\}}[\pi(k-k')]\Big/\prod_{\substack{k\in\{j+\}\\ k'\in\{i-\}}}[\pi(k-k'-\tfrac{1}{2})]\right). \tag{B22b}$$

As stated in Ref. 21, we know that our $\langle X\rangle$ is never negative. Because of this we need not keep factors of (-1) in either Eq. (B22) or (B23).

The expressions for $F_{ii'}$ and $F_{ii'jj'}$ look very impressive. However they are actually very simple in structure. For $F_{ii'}$ we obtain [see Eq. (II 2.10)]

$$F_{\sigma_i\sigma_{i'}}=F_{\mu_i\mu_{i'}}=\langle\sigma_{k_i}\sigma_{k_{i'}}\rangle, \tag{B23a}$$

$$F^2_{\sigma_i\mu_{i'}}=\pi|r_\mu-r_\sigma|\langle\sigma_{k_i}\sigma_{k_{i'}}\rangle\langle\sigma_{k_i}\sigma_{k_{i'}-1}\rangle, \tag{B23b}$$

$$F^2_{\mu_i\sigma_{i'}}=\pi|r_\sigma-r_\mu|\langle\sigma_{k_i}\sigma_{k_{i'}}\rangle\langle\sigma_{k_i-1}\sigma_{k_{i'}}\rangle, \tag{B23c}$$

so that for $|r_i-r_{i'}|$ much larger than a lattice constant,

$$F_{\sigma_i\mu_{i'}}=F_{\mu_i\sigma_{i'}}=\pi^{1/2}|r_\sigma-r_\mu|^{1/2}\langle\sigma_{k_i}\sigma_{k_{i'}}\rangle. \tag{B23d}$$

The terms $F_{ii'jj'}$ are best handled by a generalization of the "contraction" method of II, Eqs. (II 2.15)–(II 2.18).

Then we write

$$\langle D_{\gamma_1}(1)\,D_{\gamma_{1'}}(1')\cdots D_{\gamma_n}(n)D_{\gamma_{n'}}(n')\rangle$$

in terms of $F_{ii'}$ and $F_{ii'jj'}$. By setting $(n-1)'=n$, one obtains the correlation function

$$\langle D_{\gamma_1}(1)\,D_{\gamma_{1'}}(1')\cdots D_{\gamma_{n-1}}(n-1)\,D_{\gamma_{n'}}(n')\rangle.$$

But, this last correlation function can also be written directly. By equating these two expressions, we get

$$F_{ii',jj'} = F_{ij'}F_{i'j}/F_{ij}F_{i'j'} \quad . \tag{B24}$$

Notice that Eq. (II 2.18) turns out to be just a special case of our Eq. (B24).

This form for $F_{ii',jj'}$ allow us to write $\langle X\rangle$ as

$$\langle X\rangle = \left(\prod_i F_{ii'}\right)\left(\prod_j \prod_{i<j} \frac{F_{ij'}F_{i'j}}{F_{ij}F_{i'j'}}\right) \quad . \tag{B25a}$$

At this point we return to the original notation, i.e., form

$$(1),\ (1'),\ldots,(n),\ (n')$$

back to

$$1, 2, \ldots, N = 2n \quad ,$$

and introduce $p_i = (-1)^{2\Gamma i}$, which is $(+1)$ or (-1) if i is even or odd, respectively.

The correlation function takes the form

$$\langle X\rangle = \prod_{1\le i<j\le N} (F_{ij})^{-p_i p_j} \quad . \tag{B25b}$$

Finally, by considering the case, $|r_i - r_j| \gg$ lattice const,

$$F_{ij} = \begin{cases} \langle \sigma_i \sigma_j\rangle & \text{if } \gamma_i = \gamma_j \\ \pi |r_i - r_j|^{1/2} \langle \sigma_i \sigma_j\rangle = \dfrac{\text{const}}{\langle \sigma_i \sigma_j\rangle} & \text{if } \gamma_i = -\gamma_j \quad , \end{cases} \tag{B26}$$

Eq. (B25b) can be seen to be Eq. (4.9).

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[1]L. Kadanoff, Phys. Rev. 188, 859 (1969). Referred to as II in Appendix B.

[2]L. Kadanoff, Phys. Rev. Letters 23, 1430 (1969).

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[11]M. E. Fisher and A. E. Ferdinand [Phys. Rev. Letters 19, 169 (1967)] have used a technique like this one to determine "surface tensions" for the two-dimensional Ising model. See also Ref. 10.

[12]Note that μ_r is not local in the same sense as σ_r since μ_r appears to depend upon σ's infinitely far away from the point r. In particular $\langle \mu_r \mu_{r'}\rangle$ is proportional to an average of a product of terms like $e^{-2K\sigma\sigma'}$ over any path between r and r'. But this average is local in the sense that $\langle \mu_r \mu_{r'}\rangle$ depends only upon the values of the coupling constants in the neighborhood of r and r' (i.e., within one coherence length) and not upon the values of the K's throughout the entire lattice.

[13]R. Hecht, Phys. Rev. 158, 577 (1967); J. Stephenson, J. Math. Phys. 7, 1123 (1966).

[14]The symbol $\mathfrak{R}$ has been used in this definition of the distance variable to distinguish it from the distance variable $R = [x^2 + u^4 y^2]^{1/2}$ used in Ref. 9. We also employ a slightly different definition of ϵ in this work from that used in Ref. 9 in that ϵ (for this work) $= u^{-2}\epsilon$ (Ref. 9). These redefinitions provide variables which have more convenient symmetry properties when u is varied.

[15]R. Hecht, thesis (University of Illinois, 1967) (unpublished).

[16]In Ref. 2 this formalism was presented for all the operators on the x axis. Here we prefer the y axis instead, because in this form the present scheme of calculation reduces to the one in Ref. 9 for the case K_x, K_y const. Of course, the calculation is identical in any of the two axes, and the final results differ only in notation.

[17]A correlation function with an odd number of μ's (σ's) can be calculated, in general, with the use of a path joining the μ (σ) with biggest y to a μ (σ) variable at infinity.

[18]The material in this Appendix is not original. It is designed to make explicit the ideas contained in implicit form in the literature (see, e.g., Ref. 19).

[19]G. H. Wannier, Rev. Mod. Phys. 17, 50 (1945); see also C. Domb, Advan. Phys. 9, 1 (1960).

[20]This definition is identical to the one given in Ref. 9

[21]Equation (I 3.21), as well as (I 3.14) and (I 3.19) have some misprints and incorrect signs. Instead of these, we used in this work the following equations:

$$q(p_y) = \tfrac{1}{2}\{1 - [\Phi(p_y)]^{-\tau_3}\tau_2\}\, e^{\gamma(p_y)} + \tfrac{1}{2}\{1 + [\Phi(p_y)]^{-\tau_3}\tau_2\}\, e^{-\gamma(p_y)}, \tag{I3.14}$$

$$\Phi(p_y) = \left(\frac{Be^{ip_y} - 1}{B - e^{ip_y}}\ \frac{Ae^{ip_y} - 1}{A - e^{ip_y}}\right)^{1/2} \quad , \tag{I3.19}$$

$$g(j, k; j', k') = \int_0^{2\pi} \frac{dp_y}{2\pi}\ \exp[-\gamma(p_y)\,|j - j'| + ip_y(k - k')] \times \begin{cases} \tfrac{1}{2}\{1 - [\Phi(p_y)]^{-\tau_3}\tau_2\} & \text{for } j \le j' \\ \tfrac{1}{2}\{1 + [\Phi(p_y)]^{-\tau_3}\tau_2\} & \text{for } j > j' \end{cases} \quad . \tag{I3.21}$$

[22]The present calculation determines the correlation function $\langle X\rangle$, save for a multiplicative factor of a power of (-1). However, we are able to prove that *for operators on a line* and $\Gamma = 0$, $\langle X\rangle$ is never negative, by borrowing results already obtained by B. Kaufman and L. Onsager [Phys. Rev. 76, 1244 (1949)]. In their paper they calculated the average values $\langle iP_m Q_e\rangle$, $\langle Q_m Q_e\rangle$, and $\langle P_m Q_e\rangle$. The first one is nothing but

$$-\langle \sigma_m \mu_m \mu_{e+1} \sigma_e\rangle \quad \text{if } m > e;$$

or

$$+\langle \mu_{e+1} \sigma_e \sigma_m \mu_m\rangle \quad \text{if } e > m.$$

In both cases, $\langle X\rangle \ge 0$ for all temperatures [see their

Eq. (13)]. More complicated correlations can be handled by taking recourse to the factorization property of averages of b's, as explained in Hecht (Ref. 16). The other two averages $\langle Q_m Q_e \rangle$ and $\langle P_m P_e \rangle$ provide us with a cross-check of our result that correlation functions with $\Gamma \neq 0$ vanish for infinitesimal coupling strength along the line.

[23]See the discussion in Ref. 1, immediately below Eq. (II 2.6). Incidentally, this equation is misprinted; it should read

$$\Phi(p_y) = [-(Ae^{ip_y} - 1)/(A - e^{ip_y})]^{1/2}.$$

Also, the operators σ of this reference are located on the y axis, as indicated by their coordinates $(0, k)$ and not on the x axis as stated in the text.

[24]See, for instance, N. I. Achieser, *Theory of Approximation* (Ungar, New York, 1956), p. 19.

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Fluctuations and Physical Properties of the Two-Dimensional Crystal Lattice

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The properties of finite, but large, two-dimensional crystal lattices are discussed in the light of the lack of long-range order. We confirm, with qualifications, the important basic result that the susceptibility diverges below a critical temperature. The details of our previous paper on Bragg peaks in scattering from the two-dimensional lattice are presented and the behavior of the dynamic structure factor $S(\vec{k}, \omega)$ about the peaks is analyzed. The lattice is shown to produce a Mössbauer peak with a non-Lorentzian line shape but with a Mössbauer strength of the same order of magnitude as that of the three-dimensional lattice. Finally, it is argued that finite phonon lifetimes would affect our results quantitatively but not qualitatively.

I. INTRODUCTION

The subject of long-range order in various one- and two-dimensional (2-d) systems has recently become a matter of great interest. There exist 2-d systems which possess long-range order, notably the Ising and probably the anisotropic Heisenberg models. On the other hand, we have many examples of 2-d systems for which long-range order can be rigorously shown not to exist, [1] for example, the isotropic Heisenberg model, a Bose condensate, electron pair superconductivity, and a crystalline[2] lattice. However, there is an increasing number of indications[3–5] that these last systems exhibit a variety of properties not too different from those characterising the three-dimensional (3-d) ordered analogs.

The reason for the interest in 2-d systems of the last type is that one would like to understand better the connection between mathematical long-range order and physical properties. Furthermore, it may be hoped that a 2-d geometry may be a good approximation for very thin layers and films and for materials with rod structures, when interest is focused on the motion perpendicular to the axis of alignment.

The 2-d crystal offers a particularly simple example. In the harmonic approximation it admits of an exact solution, and may serve, as shown by Jancovici, [6] as an example of a system with no long-range order that still has an "infinite" susceptibility. It has also been observed[4] that this nonordered structure gives rise to Bragg-like peaks in the x-ray structure factor, reminiscent of those obtained in ordered lattice structures. The reason for this effect is that the "divergence" in the mean-square fluctuation in the position of an atom, which leads to a vanishing order parameter, is caused by long-wavelength phonons, thus not affecting the short-range order. Related to this is the fact that the correlation function falls off slowly, as $1/r^{\alpha}$, and not exponentially.

The case of x-ray scattering is particularly simple because what is observed is the integral over all frequencies of the dynamic structure factor $S(\vec{k}, \omega)$, which is related to the equal-time cor-